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DIFFERENCE SCHEMES FOR MIXED PROBLEM FOR HEAT EQUATION IN ANGULAR DOMAIN

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ABSTRACT

In the paper, it is constructed with difference schemes which approximately mixed problem for heat equation and shown their stability. There exist various methods to develop difference schemes which are mainly based on exchanging derivatives with difference schemes.

KEYWORDS: *Difference Schemes, Wave Equation, Approximation, Finite Difference, Stability, Boundary Conditions*

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INTRODUCTION

Preliminaries

Problem 1: We consider the following mixed problem for the equation below

$$u_{tt} - u_{xx} - u_{yy} = 0 \tag{1}$$

With boundary conditions for $x = 0,\ (t, y) \in R_+^2$

$$u_t - a_1 u_x - b_1 u_y = 0, \tag{2}$$

For $y = 0,\ (t, x) \in R_+^2$

$$u_t - a_2 u_x - b_2 u_y = 0, \tag{3}$$

And initial condition for $t = 0,\ \ (x, y) \in R_+^2$

$$u(0, x, y) = \varphi(x, y),\ \ u_t(0, x, y) = \psi(x, y) \tag{4}$$

In the domain $R_+^3 = \left\{ (t, x, y) \middle|\ t, x, y > 0 \right\}$.

Here $a_1, b_1, a_2, b_2 \in R$ and $r \to 0$

$$u_t = o(r^{-1/2}),\ u_x = o(r^{-1/2}), u_y = o(r^{-1/2}), r = \sqrt{x^2 + y^2}\ .$$

If for the conditions (2), (3) Shapiro–Lopatinski condition holds [17,50]

- $x=0$ in $a_1>0, |b_1|<1$;
- $y=0$ in $a_2>0, |b_2|<1$;

We rewrite the problem (1)–(4) in new coordinates system ξ, θ ($x=r\cos\theta,\ y=r\sin\theta,\ \xi=\ln r$)

$$e^{2\xi}u_{tt}-u_{\theta\theta}-u_{\xi\xi}=0, \text{ for } t>0,\ 0<\theta<\frac{\pi}{2}, \xi\in R^1 \tag{5}$$

$$e^{\xi}u_t+a_1u_\theta-b_1u_\xi=0, \text{ for } \theta=\frac{\pi}{2},\ t>0,\quad \xi\in R^1 \tag{6}$$

$$e^{\xi}u_t-a_2u_\theta-b_2u_\xi=0. \text{ for } \theta=0,\ t>0,\quad \xi\in R^1 \tag{7}$$

Then boundary conditions, $t=0,\quad 0<\theta<\frac{\pi}{2}$

$$\left.\begin{aligned} u\big|_{t=0} &= \tilde{\varphi}(\xi,\theta)=\varphi(e^{\xi}\cos\theta, e^{\xi}\sin\theta)\\ u_t\big|_{t=0} &= \tilde{\psi}(\xi,\theta)=\psi(e^{\xi}\cos\theta, e^{\xi}\sin\theta)\end{aligned}\right\} \tag{8}$$

In $|\xi|\to\infty$ an initial conditions $u_t=o(e^{-\frac{1}{2}\xi}),\ u_\theta=o(e^{-\frac{1}{2}\xi}), u_\xi=o(e^{-\frac{1}{2}\xi})$. Obtained problem (5)-(8) in the domain $t>0, 0<\theta<\frac{\pi}{2}, \xi\in R^1$ is reduced to the following mixed problem consisting of system of symmetric *t*-hyperbolic equations

$$\left\{e^{\xi}A_0\frac{\partial}{\partial t}-B_0\frac{\partial}{\partial\theta}-C_0\frac{\partial}{\partial\xi}+Q_0\right\}V=0 \tag{9}$$

Boundary conditions

$$\theta=\frac{\pi}{2},\ t>0,\quad \xi\in R^1 \text{ for } \vartheta_1+a_1\vartheta_2-b_1\vartheta_3=0, \tag{10}$$

$$\theta=0,\ t>0,\quad \xi\in R^1 \text{ for } \vartheta_1-a_2\vartheta_2-b_2\vartheta_3=0, \tag{11}$$

Initial condition, for $t=0,\quad 0<\theta<\frac{\pi}{2}, \xi\in R$ $V=\left(e^{\xi}\tilde{\psi}(\theta,\xi),\quad \tilde{\varphi}'_\theta(\theta,\xi),\quad \tilde{\varphi}'_\xi(\theta,\xi)\right)'$ and when

$$|\xi|\to\infty\ V=o(e^{\frac{1}{2}\xi}) \tag{12}$$

Here

$$A_0=\begin{pmatrix} k & l & m \\ l & k & 0 \\ m & 0 & k \end{pmatrix}, B_0=\begin{pmatrix} l & k & 0 \\ k & l & m \\ 0 & m & -l \end{pmatrix}, C_0=\begin{pmatrix} m & 0 & k \\ 0 & -m & l \\ k & l & m \end{pmatrix}, Q_0=\begin{pmatrix} m & 0 & 0 \\ 0 & 0 & 0 \\ k & 0 & 0 \end{pmatrix},$$

$$V=\begin{pmatrix} \vartheta_1 \\ \vartheta_2 \\ \vartheta_3 \end{pmatrix}=\begin{pmatrix} e^{\xi}u_t \\ u_\theta \\ u_\xi \end{pmatrix},$$

according to [8], taking notation $V=e^{\mu\xi}Y$, we get equality of dissipative energy integral

$$e^{\xi}(A_0Y,Y)_t-(B_0Y,Y)_\theta-(C_0Y,Y)_\xi+([Q_0+Q_0^*-2\mu C_0+\frac{d}{d\theta}B_0]Y,Y)=0, \tag{13}$$

And assuming $\mu=\frac{1}{2}$ as well as when $|\xi|\to\infty$ $\|Y\|^2=(Y,Y)\to 0$, we integrate it in the domain

$$\Pi=\left\{(\theta,\xi)\middle|\ \xi\in R, 0\le\theta\le\frac{\pi}{2}\right\}$$

$$\frac{d}{dt}\left\{\iint_\Pi e^{\xi}\left(A_0Y,Y\right)d\xi d\theta\right\}-\int_R\left\{\left(B_0Y,Y\right)\Big|_{\theta=\frac{\pi}{2}}-\left(B_0Y,Y\right)\Big|_{\theta=0}\right\}d\xi+$$

$$+\iint_\Pi\left(\left[Q_0+Q_0^*-C_0+\frac{d}{d\theta}B_0\right]Y,Y\right)d\xi d\theta=0$$

We use the following equalities (see [3],):

- $A_0(\theta)=T_0^*\begin{pmatrix} H(\theta) & O \\ O & H(\theta) \end{pmatrix}T_0;$
- $B_0(\theta)=T_0^*\begin{pmatrix} O & -H(\theta) \\ -H(\theta) & O \end{pmatrix}T_0;$
- $C_0(\theta)=T_0^*\begin{pmatrix} -H(\theta) & O \\ O & H(\theta) \end{pmatrix}T_0;$
- $Q_0(\theta)+Q_0^*(\theta)=\begin{pmatrix} 2m(\theta) & 0 & k(\theta) \\ 0 & 0 & 0 \\ k(\theta) & 0 & 0 \end{pmatrix}=T_0^*\begin{pmatrix} -H(\theta) & L(\theta) \\ L(\theta) & H(\theta) \end{pmatrix}T_0;$

$$O=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ L(\theta)=\begin{pmatrix} -l(\theta) & m(\theta) \\ m(\theta) & l(\theta) \end{pmatrix},$$

$$H(\theta)=\begin{pmatrix} k(\theta)-m(\theta) & -l(\theta) \\ -l(\theta) & k(\theta)+m(\theta) \end{pmatrix},$$

$$T_0=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Now we study $e^{\xi}(A_0Y,Y)$ and $\left(\left[Q_0+Q_0^*-C_0+\frac{d}{d\theta}B_0\right]Y,Y\right)$. In order that this inequality holds $e^{\xi}(A_0Y,Y)=(A_0V,V)=(HW_1,W_1)+(HW_2,W_2)>0$, it has to be $H>0$, where

$$W=\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}=\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}=T_0V$$

If $k=k(\theta)>0,\ k^2(\theta)-m^2(\theta)-l^2(\theta)>0$, then $H>0$.

$$\left(\left[Q_0+Q_0^*-C_0+\frac{d}{d\theta}B_0\right]Y,Y\right)=e^{-\xi}\left(\left[\begin{pmatrix} -H & L \\ L & H \end{pmatrix}-\begin{pmatrix} -H & O \\ O & H \end{pmatrix}+\begin{pmatrix} O & -H' \\ -H' & O \end{pmatrix}\right]V,V\right)$$

If $H'=\frac{d}{d\theta}H=L$, that's if $(k-m)'=-l,\quad (k+m)'=l,\quad -l'=m$, then square form is equal to 0. Finding the solution of these differential equations, we have

$$k(\theta)=\begin{cases} k(0) \\ k\left(\frac{\pi}{2}\right) \end{cases},\quad l(\theta)=\begin{cases} l(0)\cos\theta-m(0)\sin\theta \\ m\left(\frac{\pi}{2}\right)\cos\theta+l\left(\frac{\pi}{2}\right)\sin\theta \end{cases},\quad m(\theta)=\begin{cases} m(0)\cos\theta+l(0)\sin\theta \\ -l\left(\frac{\pi}{2}\right)\cos\theta+m\left(\frac{\pi}{2}\right)\sin\theta. \end{cases}$$

Now we consider square forms $-(B_0Y,Y)\big|_{\theta=\frac{\pi}{2}}$ and $(B_0Y,Y)\big|_{\theta=0}$.

We rewrite Boundary

$$-(B_0Y,Y)\big|_{\theta=\frac{\pi}{2}}=-e^{-\xi}(B_0V,V)\big|_{\theta=\frac{\pi}{2}}=e^{-\xi}\left\{(HW_2,W_1)+(HW_1,W_2)\right\}\big|_{\theta=\frac{\pi}{2}}$$

condition (10) as $W_1=SW_2$ when $\theta=\frac{\pi}{2}$, where

$$S=\begin{pmatrix} \frac{2a_1}{1+b_1} & -\frac{1-b_1}{1+b_1} \\ 1 & 0 \end{pmatrix}.$$

With the help of this equality, we get

$$-\left(B_0Y,Y\right)\Big|_{\theta=\frac{\pi}{2}} = e^{-\xi}\left(\left[S^*H+HS\right]W_2,W_2\right)\Big|_{\theta=\frac{\pi}{2}}$$

And analogously for $\theta=0$ rewriting boundary condition (11) as $W_1=RW_2$ we find

$$\left(B_0Y,Y\right)\Big|_{\theta=0} = -e^{-\xi}\left(\left[R^*H+HR\right]W_2,W_2\right)\Big|_{\theta=0},$$

Where

$$R=\begin{pmatrix} -\dfrac{2a_2}{1+b_2} & -\dfrac{1-b_2}{1+b_2} \\ 1 & 0 \end{pmatrix}.$$

For $n+1$, solutions are obtained by a formula explicitly while the solutions for n of the difference schemes are known, this scheme is called explicit. Even though samples of some difference schemes to be considered look like the samples of explicit schemes, they are not actually.

Explicit Right Difference Scheme: To solve problem-1 numerically, we employ explicit right difference scheme which approximates differential problem. For this, we rewrite the system (9) in the following form (7):

$$e^{\xi}A_0\frac{\partial Y}{\partial t}-\frac{\partial[B_0Y]}{\partial\theta}-C_0\frac{\partial Y}{\partial\xi}+\left[Q_0-\mu C_0+\frac{d}{d\theta}B_0\right]Y=0, \quad (14)$$

$$e^{\xi}A_0\frac{\partial Y}{\partial t}-B_0\frac{\partial Y}{\partial\theta}-C_0\frac{\partial Y}{\partial\xi}+[Q_0-\mu C_0]Y=0 \quad (15)$$

We multiply the systems (14) and (15) by $D=diag(y_1,y_2,y_3)$ from left side. Adding obtained systems, we find

$$\begin{aligned} &e^{\xi}DA_0\frac{\partial Y}{\partial t}+e^{\xi}DA_0\frac{\partial Y}{\partial t}-D\frac{\partial[B_0Y]}{\partial\theta}-DB_0\frac{\partial Y}{\partial\theta}-DC_0\frac{\partial Y}{\partial\xi}- \\ &-DC_0\frac{\partial Y}{\partial\xi}+D[Q_0-\mu C_0+\frac{d}{d\theta}B]Y+D[Q_0-\mu C_0]Y=0. \end{aligned} \quad (16)$$

In the domain $t>0,(\theta,\xi)\in\Pi$, we employ mesh with step-sizes respectively

$$\Delta t=\Delta_t,\ \Delta\theta=\Delta_\theta,\ \Delta\xi=\Delta_\xi.$$

We take the following notations and define norm as follows:

$$\begin{aligned} &Y_{ij}^n=Y(n\Delta_t,i\Delta_\theta,j\Delta_\xi)=(y_1(n\Delta_t,i\Delta_\theta,j\Delta_\xi),y_2(n\Delta_t,i\Delta_\theta,j\Delta_\xi),y_3(n\Delta_t,i\Delta_\theta,j\Delta_\xi))', \\ &i=\overline{0,I},\ n,|j|=0,1,\ldots; \end{aligned}$$

$$\left\|Y^n\right\|_{A_0}^2=\Delta_\theta\Delta_\xi\sum_{i=0}^{I-1}\sum_{j=-\infty}^{\infty}e^{\xi_j}(A_0Y_{ij}^n,Y_{ij}^n),$$

$$L = (1, 1, 1)', \ \mu = \frac{1}{2}.$$

Using the above notations, we develop difference scheme approximating (16):

$$\begin{aligned} & e^{\xi_j} D_{ij}^n (\mathrm{A}_0)_i \frac{Y_{ij}^{n+1} - Y_{ij}^n}{\Delta_t} + e^{\xi_j} D_{ij}^{n+1} (\mathrm{A}_0)_i \frac{Y_{ij}^{n+1} - Y_{ij}^n}{\Delta_t} - D_{ij}^n \frac{(\mathrm{B}_0 Y)_{i+1j}^n - (\mathrm{B}_0 Y)_{ij}^n}{\Delta_\theta} - \\ & - D_{i+1j}^n (\mathrm{B}_0)_{i+1} \frac{Y_{i+1j}^n - Y_{ij}^n}{\Delta_\theta} - D_{ij}^n (\mathrm{C}_0)_i \frac{Y_{ij+1}^n - Y_{ij}^n}{\Delta_\xi} - D_{ij+1}^n (\mathrm{C}_0)_i \frac{Y_{ij+1}^n - Y_{ij}^n}{\Delta_\xi} + \\ & + D_{ij}^n \left[2\mathrm{Q}_0 - 2\mu \mathrm{C}_0 + \frac{d}{d\theta} \mathrm{B}_0 \right]_i Y_{ij}^n = 0, \ \mathrm{i} = \overline{0, \mathrm{I} - 1}, \ |j| = 0,1,2,.., \ n = \overline{0, N-1}, \end{aligned} \tag{17}$$

for $i = 0$, $|j| = 0,1,2,\ldots$

$$(y_1)_{0j}^n - a_2 (y_2)_{0j}^n - b_2 (y_3)_{0j}^n = 0, \tag{18}$$

for $i = I$, $|j| = 0,1,2,\ldots$

$$(y_1)_{Ij}^n + a_1 (y_2)_{Ij}^n - b_1 (y_3)_{Ij}^n = 0, \tag{19}$$

for $n = 0$, $i = 0,1,\ldots,I$, $|j| = 0,1,2,\ldots$

$$Y_{ij}^0 = \left(e^{\frac{1}{2}\xi_j} \tilde{\psi}(\xi_j, \theta_i), e^{\frac{1}{2}\xi_j} \tilde{\varphi}'_\theta(\xi_j, \theta_i), e^{\frac{1}{2}\xi_j} \tilde{\varphi}'_\xi(\xi_j, \theta_i) \right)'. \tag{20}$$

Sample of this difference scheme as shown in figure-1.1, consists of the systems of equations which are not linear. It is easily seen that, this scheme approximates differential equation with first order. The boundary conditions are precisely approximated. As we mentioned above to compute schemes approximation error exact solution is calculated by the scheme (17) and we denote this error δf_h as the norm of the vector $E(t_n, \theta_i, \xi_j)$. Approximation of difference scheme for the sample equation is shown in the figures 2-4 and table-1. Here $t_{10} = 0.3$, $t_{20} = 0.6$, $t_{30} = 0.9$, $\theta_2 = 0.524$, $\xi_6 = 3.2$, $\max_{i,j}(E_{10}) = 7.015 \times 10^{-7}$, $\max_{i,j}(E_{20}) = 5.729 \times 10^{-7}$, $\max_{i,j}(E_{30}) = 4.719 \times 10^{-7}$.

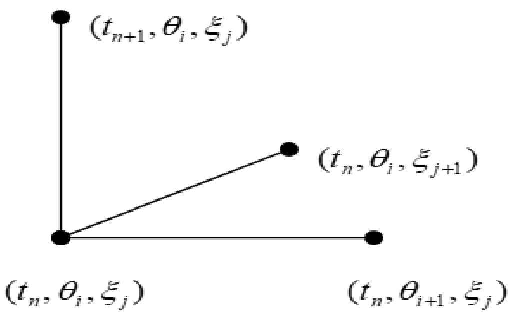

Figure 1: Difference Scheme.

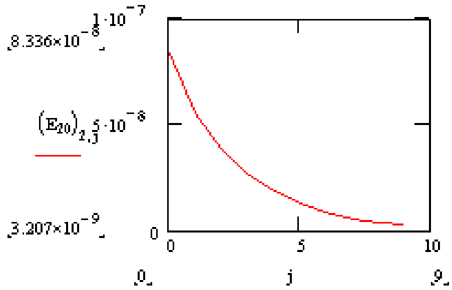

Figure 2: Values At $t_{20} = 0.6, \quad \theta_2 = 0.524$ **of the Error** $(E_n)_{i\,j}$ **of the Sample Problem.**

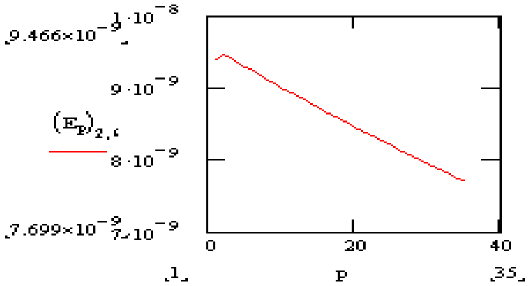

Figure 3: Values At $\theta_2 = 0.524, \xi_6 = 3.2$.

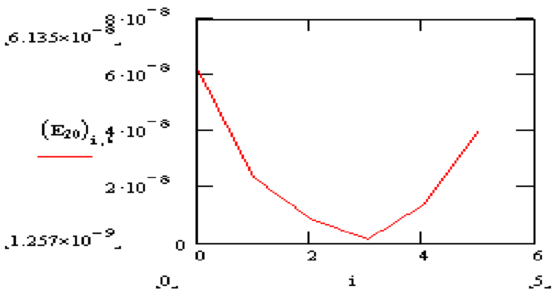

Figure 4:. Values At $t_{20} = 0.6, \xi_6 = 3.2$ **of the Error** $(E_n)_{i\,j}$ **of the Error** $(E_n)_{i\,j}$.

Table 1: Value At $t_{30} = 0.9$ $(E_n)_{i\,j}$ **of Approximation Error**

	I=0	I=1	I=2	I=3	I=4	I=5
j=0	4.719e-7	1.911e-7	6.868e-8	1.762e-8	1.244e-7	3.42e-7
j=1	3.457e-7	1.367e-7	4.735e-8	1.546e-8	9.432e-8	2.455e-7
j=2	2.482e-7	9.721e-8	3.410e-8	9.628e-9	6.457e-8	1.734e-7
j=3	1.755e-7	6.815e-8	2.416e-8	5.942e-9	4.378e-8	1.208e-7
j=4	1.224e-7	4.718e-8	1.687e-8	3.638e-9	2.943e-8	8.310e-8
j=5	8.437e-8	3.231e-8	1.164e-8	2.213e-9	1.963e-8	5.659e-8
j=6	5.754e-8	2.191e-8	7.942e-9	1.337e-9	1.301e-8	3.818e-8
j=7	3.889e-8	1.473e-8	5.369e-9	8.037e-10	8.572e-9	2.555e-8
j=8	2.607e-8	9.832e-9	3.601e-9	4.802e-10	5.616e-9	1.698e-8
j=9	1.955e-8	7.938e-9	3.517e-9	7.528e-10	2.541e-9	9.788e-9

Theorem 1: If for (18)–(19) it holds Shapiro–Lopatinski condition, *that's if* $a_1 > 0, |b_1| < 1$ any $a_2 > 0, |b_2| < 1$, *then difference scheme* (17)-(20) *is stable with respect to* $\sqrt{J^n}$ *energetic norm, here* $J^n = \Delta_\theta \Delta_\xi \sum_{i=0}^{I-1} \sum_{j=-\infty}^{+\infty} (A_0 V, V)_{ij}^n$.

Proof. We multiply (17) by vector L and for convenience, we perform this operation for each variable separately:

- $$\left(D_{ij}^n (A_0)_i \frac{Y_{ij}^{n+1} - Y_{ij}^n}{\Delta_t}, L\right) + \left(D_{ij}^{n+1} (A_0)_i \frac{Y_{ij}^{n+1} - Y_{ij}^n}{\Delta_t}, L\right) = \frac{1}{\Delta_t}\left((A_0)_i \left(Y_{ij}^{n+1} - Y_{ij}^n\right), (DL)_{ij}^n\right) +$$
$$+\frac{1}{\Delta_t}\left(Y_{ij}^{n+1} - Y_{ij}^n, (A_0)_i (DL)_{ij}^{n+1}\right) = \frac{1}{\Delta_t}\left([A_0 Y]_{ij}^{n+1} - [A_0 Y]_{ij}^n, Y_{ij}^n\right) + \frac{1}{\Delta_t}\left(Y_{ij}^{n+1} - Y_{ij}^n, [A_0 Y]_{ij}^{n+1}\right) =$$
$$= \frac{1}{\Delta_t}\left\{\left((A_0)_i \left(Y_{ij}^{n+1} - Y_{ij}^n\right), Y_{ij}^n\right) + \left((A_0)_i \left(Y_{ij}^{n+1} - Y_{ij}^n\right), Y_{ij}^{n+1}\right)\right\} =$$
$$= \frac{1}{\Delta_t}\left\{\left((A_0)_i Y_{ij}^{n+1}, Y_{ij}^n\right) - \left((A_0)_i Y_{ij}^n, Y_{ij}^n\right) + \left((A_0)_i Y_{ij}^{n+1}, Y_{ij}^{n+1}\right) - \left((A_0)_i Y_{ij}^n, Y_{ij}^{n+1}\right)\right\} =$$
$$= \frac{1}{\Delta_t}\left((A_0)_i Y_{ij}^{n+1}, Y_{ij}^{n+1}\right) - \frac{1}{\Delta_t}\left((A_0)_i Y_{ij}^n, Y_{ij}^n\right) = \frac{1}{\Delta_t}(A_0 Y, Y)_{ij}^{n+1} - \frac{1}{\Delta_t}(A_0 Y, Y)_{ij}^n;$$

Here we used $D_{ij}^n L = Y_{ij}^n$, $D_{ij}^{n+1} L = Y_{ij}^{n+1}$ and $A_0 = A_0^*$.

- $$\left(D_{ij}^n \frac{[B_0 Y]_{i+1j}^n - [B_0 Y]_{ij}^n}{\Delta_\theta}, L\right) + \left(D_{i+1j}^n [B_0]_{i+1} \frac{Y_{i+1j}^n - Y_{ij}^n}{\Delta_\theta}, L\right) = \frac{1}{\Delta_\theta}\left([B_0 Y]_{i+1j}^n - [B_0 Y]_{ij}^n, Y_{ij}^n\right) +$$
$$+\frac{1}{\Delta_\theta}\left([B_0 Y]_{i+1j}^n, Y_{i+1j}^n - Y_{ij}^n\right) = \frac{1}{\Delta_\theta}\left(\left((B_0)_{i+1} Y_{i+1j}^n, Y_{ij}^n\right) - \left((B_0)_i Y_{ij}^n, Y_{ij}^n\right) + \left((B_0)_{i+1} Y_{i+1j}^n, Y_{i+1j}^n\right) -\right.$$
$$\left.- \left((B_0)_{i+1} Y_{ij}^n, Y_{i+1j}^n\right)\right) = \frac{1}{\Delta_\theta}(B_0 Y, Y)_{i+1j}^n - \frac{1}{\Delta_\theta}(B_0 Y, Y)_{ij}^n;$$

- $$\left(D_{ij}^n (C_0)_i \frac{Y_{ij+1}^n - Y_{ij}^n}{\Delta_\xi}, L\right) + \left(D_{ij+1}^n (C_0)_i \frac{Y_{ij+1}^n - Y_{ij}^n}{\Delta_\xi}, L\right) = \frac{1}{\Delta_\xi}\left(\left((C_0)_i Y_{ij+1}^n, D_{ij}^n L\right) - \left((C_0)_i Y_{ij}^n, D_{ij}^n L\right) +\right.$$
$$\left.+ \left((C_0)_i Y_{ij+1}^n, D_{ij+1}^n L\right) - \left((C_0)_i Y_{ij}^n, D_{ij+1}^n L\right)\right) = \frac{1}{\Delta_\xi}(C_0 Y, Y)_{ij+1}^n - \frac{1}{\Delta_\xi}(C_0 Y, Y)_{ij}^n;$$

- $$\left(D_{ij}^n \left[2Q_0 - C_0 + \frac{d}{d\theta} B_0\right]_i Y_{ij}^n, L\right) = \left(\left[2Q_0 - C_0 + \frac{d}{d\theta} B_0\right]_i Y_{ij}^n, D_{ij}^n L\right) =$$
$$= \left(\left[2Q_0 - C_0 + \frac{d}{d\theta} B_0\right]_i Y_{ij}^n, Y_{ij}^n\right) = \left(\left[Q_0 + Q_0^* - C_0 + \frac{d}{d\theta} B_0\right] Y_{ij}^n, Y_{ij}^n\right)$$

here it is employed $\left(Q_0 Y_{ij}^n, Y_{ij}^n\right) = \frac{1}{2}\left(\left(Q_0 Y_{ij}^n, Y_{ij}^n\right) + \left(Q_0^* Y_{ij}^n, Y_{ij}^n\right)\right)$.

Summing up above equalities, we get discreet analogue of differential representation of energy integral:

$$e^{\xi_j}\frac{1}{\Delta_t}\left\{(A_0Y,Y)_{ij}^{n+1}-(A_0Y,Y)_{ij}^{n}\right\}-\frac{1}{\Delta_\theta}\left\{(B_0Y,Y)_{i+1j}^{n}-(B_0Y,Y)_{ij}^{n}\right\}-$$
$$-\frac{1}{\Delta_\xi}\left\{(C_0Y,Y)_{ij+1}^{n}-(C_0Y,Y)_{ij}^{n}\right\}+\left(\left[Q_0+Q_0^*-C_0+\frac{d}{d\theta}B_0\right]Y,Y\right)_{ij}^{n}=0 \quad (21)$$

Multiplying both sides of (21) by Δ_ξ,Δ_θ, we sum up in i from 0 to $I-1$, in j from $-\infty$ to $+\infty$ and using $\|Y^n\|=(Y_{ij}^n,Y_{ij}^n)^{1/2}\to 0$ when $|\xi|\to\infty$, we have

$$\|Y^{n+1}\|_{A_0}^2-\|Y^n\|_{A_0}^2=$$
$$=\Delta_\theta\cdot\Delta_\xi\cdot\sum_{j=-\infty}^{+\infty}\left\{(B_0Y,Y)_{Ij}^{n}-(B_0Y,Y)_{0j}^{n}\right\}-\sum_{i=0}^{I-1}\sum_{j=-\infty}^{\infty}\left(\left[Q_0+Q_0^*-C_0+\frac{d}{d\theta}B_0\right]Y,Y\right)_{ij}^{n}$$

One can easily check that the equalities hold:

$$A_0(\theta_i)=T_0^*\begin{pmatrix}H(\theta_i) & O\\ O & H(\theta_i)\end{pmatrix}T_0,\quad B_0(\theta_i)=T_0^*\begin{pmatrix}O & -H(\theta_i)\\ -H(\theta_i) & O\end{pmatrix}T_0,$$

$$C_0(\theta_i)=T_0^*\begin{pmatrix}-H(\theta_i) & O\\ O & H(\theta_i)\end{pmatrix}T_0,\quad O=\begin{pmatrix}0&0\\0&0\end{pmatrix},\quad L(\theta_i)=\begin{pmatrix}-l(\theta_i) & m(\theta_i)\\ m(\theta_i) & l(\theta_i)\end{pmatrix},\quad H(\theta_i)=\begin{pmatrix}k(\theta_i)-m(\theta_i) & -l(\theta_i)\\ -l(\theta_i) & k(\theta_i)+m(\theta_i)\end{pmatrix},$$

$$Q_0(\theta_i)+Q_0^*(\theta_i)=\begin{pmatrix}2m(\theta_i) & 0 & k(\theta_i)\\ 0&0&0\\ k(\theta_i)&0&0\end{pmatrix}=T_0^*\begin{pmatrix}-H(\theta_i) & L(\theta_i)\\ L(\theta_i) & H(\theta_i)\end{pmatrix}T_0,$$

$$T_0=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&-1\\0&-1&0\\0&-1&0\\1&0&1\end{pmatrix},\qquad \theta_i=i\Delta_\theta, i=0,1,\dots,I\,.$$

According to the above

$$e^{\xi_j}\left((A_0)_iY_{ij}^n,Y_{ij}^n\right)=e^{\xi_j}(A_0Y,Y)_{ij}^n=e^{(1-2\mu)\xi_j}(A_0\,V,\,V)_{ij}^n=$$
$$=e^{(1-2\mu)\xi_j}\left\{(H(\theta)W_1,W_1)_{ij}^n+(H(\theta)W_2,W_2)_{ij}^n\right\},$$

if $k(\theta_i)>0,\ k^2(\theta_i)-m^2(\theta_i)-l^2(\theta_i)>0,\ i=\overline{0,I}$, then $H(\theta_i)>0$ and hence

$$e^{\xi_j}\left((A_0)_iY_{ij}^n,Y_{ij}^n\right)>0\,, \quad (22)$$

where $W_{ij}^n=T_0V_{ij}^n$.

If

$$\frac{d}{d\theta}H(\theta)=\begin{pmatrix} k'(\theta)-m'(\theta) & -l'(\theta) \\ -l'(\theta) & k'(\theta)+m'(\theta)\end{pmatrix}=\begin{pmatrix} -l(\theta) & m(\theta) \\ m(\theta) & l(\theta)\end{pmatrix}=L(\theta)$$

then

$$\left(\left[Q_0+Q_0^*-C_0+\frac{d}{d\theta}B_0\right]_i Y_{ij}^n, Y_{ij}^n\right)=\left(\left[\begin{pmatrix} -H(\theta_i) & L(\theta_i) \\ L(\theta_i) & H(\theta_i)\end{pmatrix}-\right.\right.$$
$$\left.\left.-\begin{pmatrix} -H(\theta_i) & O \\ O & H(\theta_i)\end{pmatrix}+\begin{pmatrix} O & -H'(\theta_i) \\ -H'(\theta_i) & O\end{pmatrix}\right]W_{ij}^n, W_{ij}^n\right)=0. \tag{23}$$

Hence we find $\left(k(\theta)-m(\theta)\right)'=-l(\theta),\quad \left(k(\theta)+m(\theta)\right)'=l(\theta),\quad -l'(\theta)=m(\theta)$ or $k'(\theta)=0,\quad l'(\theta)=-m(\theta),\quad m'(\theta)=l(\theta)$. Solving these differential equations, we obtain

$$k(\theta_i)=\begin{cases} k(\theta_0) \\ k(\theta_I)\end{cases},\quad l(\theta_i)=\begin{cases} l(\theta_0)\cos\theta_i-m(\theta_0)\sin\theta_i \\ m(\theta_I)\cos\theta_i+l(\theta_I)\sin\theta_i\end{cases},\quad m(\theta_i)=\begin{cases} m(\theta_0)\cos\theta_i+l(\theta_0)\sin\theta_i \\ -l(\theta_I)\cos\theta_i+m(\theta_I)\sin\theta_i.\end{cases}$$

$$-\left(B_0Y,Y\right)_{Ij}^n=-\left((B_0)_I e^{-\frac{1}{2}\xi_j}V_{Ij}^n, e^{-\frac{1}{2}\xi_j}V_{Ij}^n\right)=-e^{-\xi_j}\left(B_0V,V\right)_{Ij}^n=$$
$$=-e^{-\xi_j}\left(T_0^*\begin{pmatrix} O & -H(\frac{\pi}{2}) \\ -H(\frac{\pi}{2}) & O\end{pmatrix}T_0V,V\right)_{Ij}^n=e^{-\xi_j}\left\{\left(H(\tfrac{\pi}{2})W_2,W_1\right)_{Ij}^n+\left(H(\tfrac{\pi}{2})W_1,W_2\right)_{Ij}^n\right\}$$

We rewrite boundary condition (19) as $\left(W_1\right)_{Ij}^n=S\left(W_2\right)_{Ij}^n$, where

$$S=\begin{pmatrix} \frac{2a_1}{1+b_1} & -\frac{1-b_1}{1+b_1} \\ 1 & 0\end{pmatrix}.$$

Due to this equality, it holds

$$-\left(B_0Y,Y\right)_{Ij}^n=e^{-\xi_j}\left(\left[S^*H(\tfrac{\pi}{2})+H(\tfrac{\pi}{2})S\right]W_2,W_2\right)_{Ij}^n.$$

According to lemma D.L. Tkachev, M.V. Gomolina ([7]) if for boundary conditions $a_1>0$ and $|b_1|<1$, $\left[S^*H(\frac{\pi}{2})+H(\frac{\pi}{2})S\right]_{i=I}>0$

From this it yields

$$-\left(B_0Y,Y\right)_{Ij}^n\geq 0. \tag{24}$$

Analogously we rewrite $\left(B_0Y,Y\right)_{0j}^n$ taking into account that boundary condition (18) is $\left(W_1\right)_{0j}^n=R\left(W_2\right)_{0j}^n$ as $\left(B_0Y,Y\right)_{0j}^n=-e^{-\xi_j}\left(\left[R^*H(0)+H(0)R\right]W_2,W_2\right)_{0j}^n$, where $R=\begin{pmatrix} -\frac{2a_2}{1+b_2} & -\frac{1-b_2}{1+b_2} \\ 1 & 0 \end{pmatrix}$. If $a_2>0$ and $|b_2|<1$ for boundary conditions, then according to Lyuapunov theorem $\left[R^*H(0)+H(0)R\right]_{i=0}<0$. Consequently, it yields

$$\left(B_0Y,Y\right)_{0j}^{n}\geq 0. \tag{25}$$

According to (22)-(25), we have the following energetic estimate

$$\left\|Y^{n+1}\right\|_{A_0}^{2}\leq\left\|Y^{n}\right\|_{A_0}^{2}.$$

From $Y_{ij}^{n}=e^{-\mu\xi_j}V_{ij}^{n}$ it is easy to understand that

$$\Delta_\theta\Delta_\xi\sum_{i=0}^{I-1}\sum_{j=-\infty}^{\infty}e^{\xi_j}(A_0Y,Y)_{ij}^{n+1}=\Delta_\theta\Delta_\xi\sum_{i=0}^{I-1}\sum_{j=-\infty}^{\infty}e^{\xi_j}\left(\left(A_0\right)_i e^{-\frac{1}{2}\xi_j}V_{ij}^{n+1},e^{-\frac{1}{2}\xi_j}V_{ij}^{n+1}\right)=$$

$$=\Delta_\theta\Delta_\xi\sum_{i=0}^{I-1}\sum_{j=-\infty}^{\infty}\left(\left(A_0\right)_i V_{ij}^{n+1},V_{ij}^{n+1}\right)=\Delta_\theta\Delta_\xi\sum_{i=0}^{I-1}\sum_{j=-\infty}^{\infty}(A_0V,V)_{ij}^{n+1}.$$

Therefore, it holds

$$J^n=\Delta_\theta\Delta_\xi\sum_{i=0}^{I-1}\sum_{j=-\infty}^{\infty}(A_0V,V)_{ij}^{n}\leq\Delta_\theta\Delta_\xi\sum_{i=0}^{I-1}\sum_{j=-\infty}^{\infty}(A_0V,V)_{ij}^{0}=J^0 \tag{26}$$

Theorem is proved.

To solve the first problem, we make use the following difference scheme:

$$\frac{U_{ij}^{n+1}-U_{ij}^{n}}{\Delta_t}-\left(U_t\right)_{ij}^{n}=0. \tag{27}$$

The above difference scheme is obtained using $U_t-U_t\equiv 0$. Now we multiply (27) by the vector $2U_{ij}^{n+1}$:

$$2\left(U_{ij}^{n+1},U_{ij}^{n+1}\right)-2\left(U_{ij}^{n},U_{ij}^{n+1}\right)-2\Delta_t\left(\left(U_t\right)_{ij}^{n},U_{ij}^{n+1}\right)=0$$

and from this equality, we have

$$\left(U_{ij}^{n+1},U_{ij}^{n+1}\right)-\left(U_{ij}^{n},U_{ij}^{n}\right)-\Delta_t\left(\left(U_t\right)_{ij}^{n},\left(U_t\right)_{ij}^{n}\right)-\Delta_t\left(U_{ij}^{n+1},U_{ij}^{n+1}\right)\leq 0.$$

Here Cauchy–Bunyakovsky $2\left(U,V\right)\leq\left(U,U\right)+\left(V,V\right)$ is employed. Obtained inequality is multiplied by $\Delta_\theta\Delta_\xi$ and is summed up with respect i from 0 till $I-1$, with respect to j from $-\infty$ till $+\infty$:

$$\Delta_\theta\Delta_\xi\sum_{i=0}^{I-1}\sum_{j=-\infty}^{\infty}\left(U_{ij}^{n+1},U_{ij}^{n+1}\right)\le\Delta_\theta\Delta_\xi\sum_{i=0}^{I-1}\sum_{j=-\infty}^{\infty}\left(U_{ij}^{n},U_{ij}^{n}\right)+$$
$$+\Delta_t\Delta_\theta\Delta_\xi\sum_{i=0}^{I-1}\sum_{j=-\infty}^{\infty}\left\{\left((U_t)_{ij}^n,(U_t)_{ij}^n\right)+\left(U_{ij}^{n+1},U_{ij}^{n+1}\right)\right\}.$$

We rewrite above inequality

$$\left\|U^{n+1}\right\|_{A_0}\le\left\|U^{n}\right\|_{A_0}+\Delta_t\left\|(U_t)^{n+1}\right\|_{A_0}+\Delta_t\left\|U^{n+1}\right\|_{A_0}. \quad (28)$$

Writing (26) as

$$C_1\left\{\Delta_\theta\Delta_\xi\sum_{i=0}^{I-1}\sum_{j=-\infty}^{+\infty}(U_t,U_t)_{ij}^{n+1}+(U_x,U_x)_{ij}^{n+1}+\left(U_y,U_y\right)_{ij}^{n+1}\right\}\le J^n\le$$
$$\le C_2\left\{\Delta_\theta\Delta_\xi\sum_{i=0}^{I-1}\sum_{j=-\infty}^{+\infty}(U_t,U_t)_{ij}^{n}+(U_x,U_x)_{ij}^{n}+\left(U_y,U_y\right)_{ij}^{n}\right\}$$

we add it to (28). It yields discreet analogue of (0.10)

$$\left\|U^{n+1}\right\|_{A_1}^2\le const\left\|U^{n}\right\|_{A_1}^2$$

where $\left\|U^{n}\right\|_{A_1}^2=\Delta_\theta\Delta_\xi\sum_{i=0}^{I-1}\sum_{j=-\infty}^{+\infty}\left\{(U,U)_{ij}^n+(U_t,U_t)_{ij}^n+(U_x,U_x)_{ij}^n+\left(U_y,U_y\right)_{ij}^n\right\}.$

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